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Lec-1. Date 1-11.

X $\underline{\mathcal{C} \subseteq \mathcal{P}(X)}$

(i) $\phi, X \in \mathcal{C}$.

(ii) $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$

(iii) $A \in \mathcal{C} \Rightarrow A^c = \bigcup_{i=1}^{n} C_i$,

$\underline{C_i \in \mathcal{C}}$, $\underline{C_i \cap C_j} = \underline{\phi}$ for $i \neq j$

pairwise disjoint

$\mathcal{C}$ is called a semi-algebra of subsets of X

Examples

(1) $X \neq \phi$, $\underline{\mathcal{C} = \mathcal{P}(X)}$

$\downarrow$

Powerset of X

$\|$

All subsets of X

$\phi, X \in \mathcal{C}$

$A \cap B \in \mathcal{C}$ if $A, B \in \mathcal{C}$

$A \in \mathcal{C} \implies A^c \in \mathcal{C}$

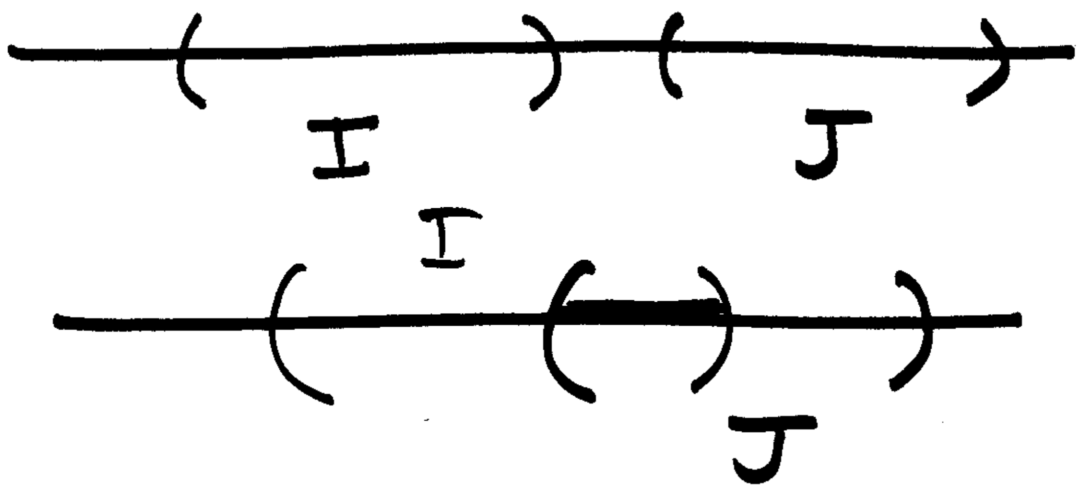
$A^c \subseteq X$, $A^c \in \mathcal{P}(X) = \mathcal{C}$

(ii) $X = \mathbb{R}$

$\mathcal{C}$ = All intervals in $\mathbb{R}$

(i) $\phi \in \mathcal{C}$, $\phi = \underbrace{(a,a)}_{\in \mathcal{C}}$ for $a \in \mathbb{R}$

(ii) $I, J \in \mathcal{C} \Longrightarrow I \cap J \in \mathcal{C}$?



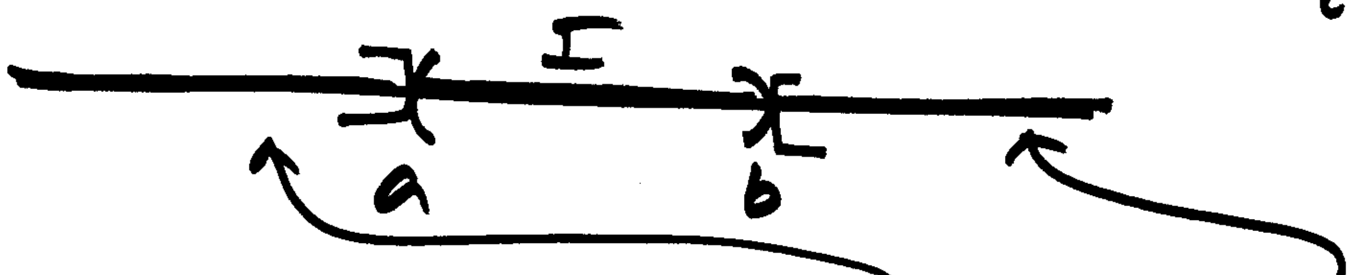
$I \cap J = \phi \in \mathcal{C}$

$I \cap J \in \mathcal{C}$

(iii) $I \in \mathcal{C}$

$$\Rightarrow \quad I^c = \bigcup_{i=1}^{n} C_i \,, \qquad C_i \in \mathcal{C}$$

$$C_i \cap C_j = \phi \text{ for } i \neq j$$



$$I^c = \mathbb{R} \setminus (a,b) = \underline{(-\infty, a]} \cup \underline{[b, +\infty)}$$

$$\mathbb{R} \setminus [a,b) = (-\infty, a) \cup [b, +\infty)$$

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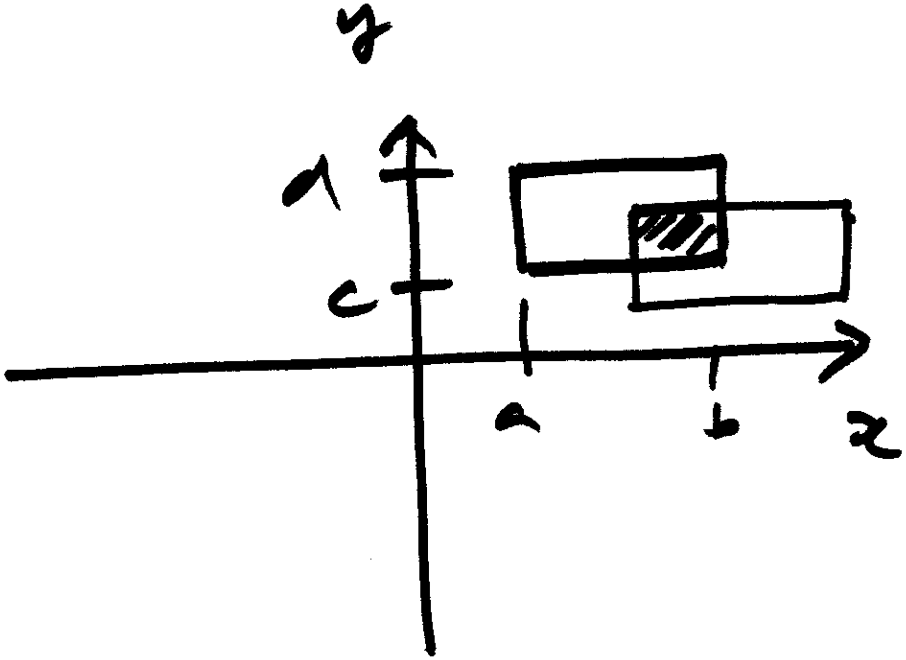
$\mathcal{C}$ the collection of all intervals in $\mathbb{R}$ is a semi-algebra of subsets of $\mathbb{R}$.

(IV)

$$X = \mathbb{R}^2$$

$\mathcal{C}$ = The collection of all rectangles in $\mathbb{R}^2$



$\underline{\phi} = (a,a) \times (a,a)$

$= (a,a) \times [c,d]$

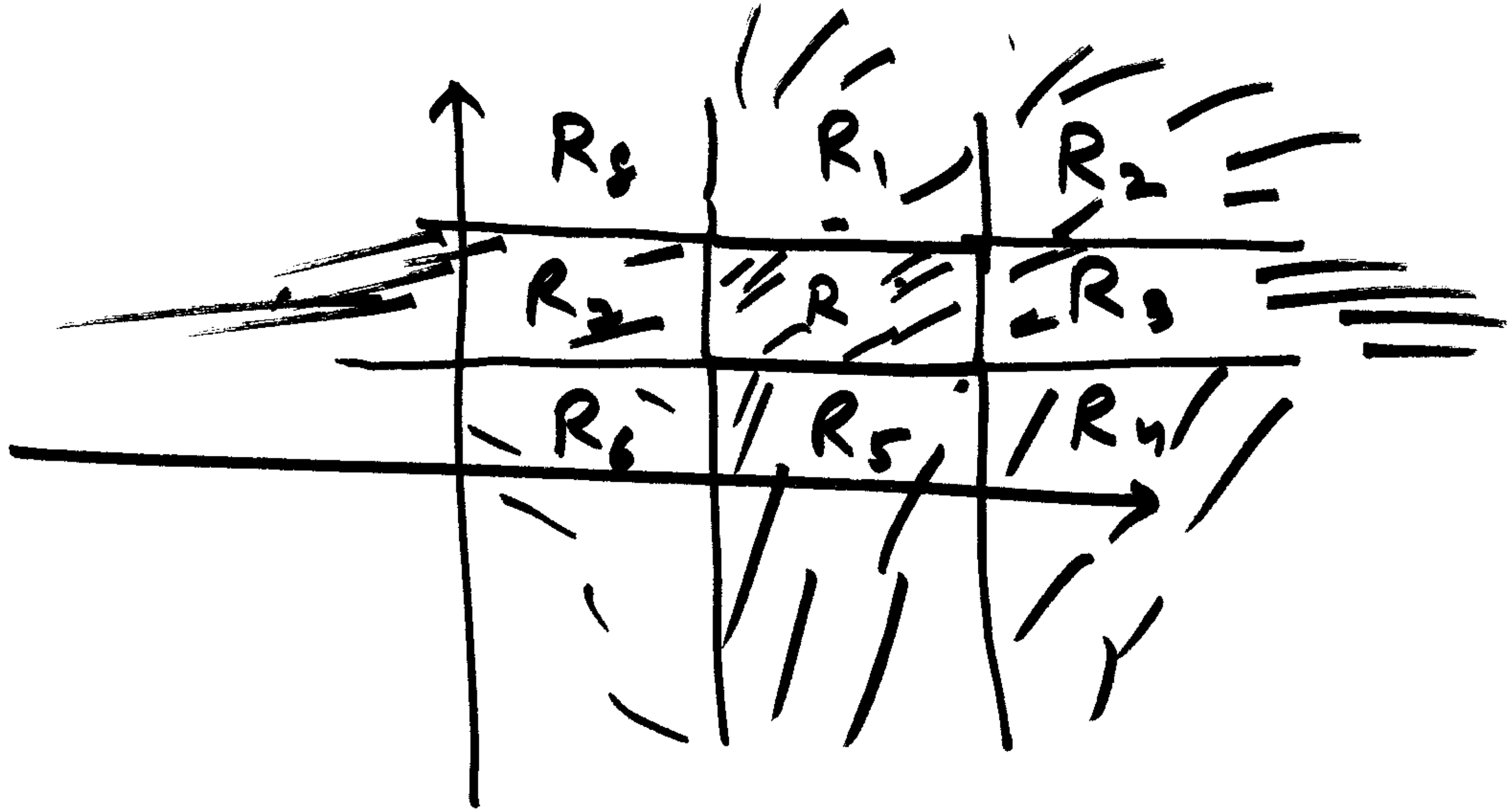
$\phi \in \mathcal{C}$

$\underline{X} = \mathbb{R}^2 = \underline{\mathbb{R}} \times \underline{\mathbb{R}}$

$\mathbb{R}$ is an interval $(-\infty, +\infty)$

R_1 , R_2

$R_1 \cap R_2$ = is a rectangle



$$R^c = \bigsqcup_{i=1}^{8} R_i \,, \quad R_i \cap R_j = \phi$$

$\mathcal{C}$ of all rectangles in $\mathbb{R}^2$
is also a semi-algebra

X — a nonempty set

$\mathcal{F} \subseteq \mathcal{P}(X)$

(i) $\phi, X \in \mathcal{F}$

(ii) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$

(iii) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$

$\mathcal{F}$ is an <u>algebra</u> of subsets of X

(i) Every algebra is also a semi-algebra

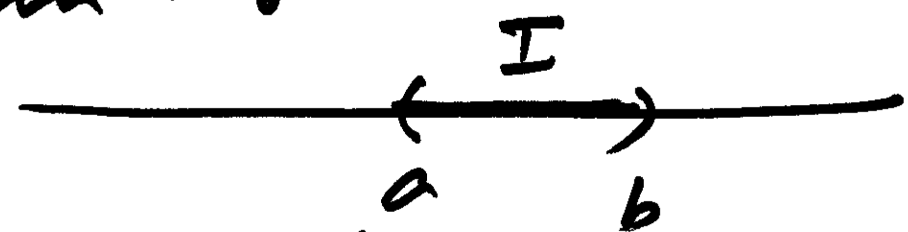
(ii) Note : $X = \mathbb{R}$

$\mathcal{C}$ = class of all intervals

$\mathcal{C}$ is a semi-algebra

Is $\mathcal{C}$ an algebra?

Not an algebra



$I \in \mathcal{C}$, I^c is not an interval

$I^c = (-\infty, a] \cup [b, +\infty)$

$\mathcal{C}$ is not an algebra

$E_1 \cap E_2 \in \mathcal{C}$?

$$(E_1 \cap E_2)^c = E_1^c \cup E_2^c$$

$$= \underbrace{(I_1^1 \cup I_2^1) \cup (J_1^1 \cup J_1^2)}$$

(iii) $\cancel{E \in \mathcal{C}}$ $E \in \mathcal{F}$

$$\Longrightarrow E^c \in \mathcal{F}.$$

$E \in \mathcal{F}$, $E^c = \cancel{E \in} \; \underline{I \cup J}$

Examples

$$X = \mathbb{R}$$

$$\mathcal{C} = \text{All intervals}$$

$$\cancel{\mathcal{F} = \left\{ E \subset \mathbb{R} \;\middle|\; E^c = \bigcup_{i=1}^{2} I_i \,,\; I_i \cap I_j = \phi \right\}}$$

Claim $\mathcal{F}$ is an algebra of subsets of $\mathbb{R}$

(i) $\mathcal{C} \subseteq \mathcal{F}$

$\Rightarrow \phi, \mathbb{R} \in \mathcal{F}$

(ii) $E_1, E_2 \in \mathcal{C}$

$E_1 \cap E_2 \in \mathcal{C}$?

$$E_1 \in \mathcal{C} \Longrightarrow \quad E_1^c = \bigsqcup_{i=1}^{2} I_i' = (I_1' \cup I_2')$$

$$E_2^c = \quad J_1' \cup J_2'$$

$$X = \mathbb{R}$$

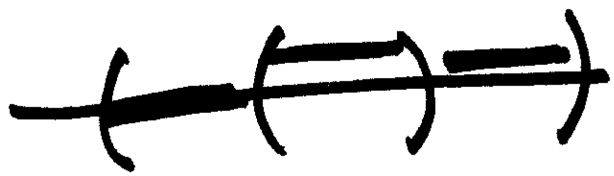
$$\mathcal{F} = \{ E \subseteq \mathbb{R} \mid E^c = \bigsqcup_{j=1}^{n} I_j \,,\ I_j \in \mathcal{C} \}$$

(i) $\mathcal{C} \subseteq \mathcal{F}$

$\Longrightarrow \emptyset, \mathbb{R} \in \mathcal{F}$

(ii) $E, F \in \mathcal{F}$, $E_1 \cup E_2 = \left(\bigsqcup_{j=1}^{n} I_j \right) \cup \left(\bigsqcup_{k=1}^{m} J_k \right)$

$$= \bigsqcup_{j=1}^{n} \bigsqcup_{k=1}^{m} \left(I_j \cap J_k \right)$$



$E, F \in \mathcal{F} \Longrightarrow E \cup F \in \mathcal{F}$

(iii) $E \in \mathcal{F} \Rightarrow E^c = \bigsqcup_{j=1}^{\infty} I_j \in \mathcal{F}$

$$E, F \in \mathcal{F} \implies E \cup F \in \mathcal{F}$$

$$E \in \mathcal{F} \implies E^c \in \mathcal{F}$$

$$\implies E \cap F = (E^c)^c \cap (F^c)^c$$

$$= (E^c \cup F^c)^c \in \mathcal{F}$$

If $\mathcal{F}$ is an algebra

then $\left.\begin{array}{l}\text{Unions} \\ \text{Complements}\end{array}\right\} \equiv \left\{\begin{array}{l}\text{Intersections} \\ \text{and} \\ \text{Complements}\end{array}\right.$